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DYNAMIC STIFFNESS ANALYSIS FOR TORSIONAL VIBRATION OF CONTINUOUS BEAMS WITH THIN-WALLED CROSS-SECTION

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An analytical method for determining natural frequencies and mode shapes of the torsional vibration of continuous beams with thin-walled cross-section is developed by using a general solution of the differential equation of motion based on Vlasov's beam theory. This method takes into account the effect of warping stiffness; it leads to an exact solution and is called the continuous mass method. Also, the approximate method based on the finite discrete element approach is presented. The mathematical relationship between the exact and the approximate methods is discussed, and the accuracy of the natural frequencies obtained by these analytical methods is investigated. Some typical continuous beams are analyzed to illustrate the applicability of the lumped, consistent, and continuous mass methods, and the computed results are given in tabular form.

1. INTRODUCTION

The computation of natural frequencies and mode shapes is a significant problem in the dynamic analysis of structures, and it is of importance in the design of structures subjected to vibratory loading caused by such agents as winds and moving vehicles. Torsional vibration in a continuous beam may be produced by aerodynamic forces and unsymmetric traffic loads. It is generally considered that the structural response of continuous beams is dependent on both lower and higher modes. In designing these kinds of continuous beams, therefore, it is essential that the natural frequencies and mode shapes be determined accurately.

From the computational point of view, the system of co-ordinates due to mass models is divided into two basic types: the discrete co-ordinate system and the distributed co-ordinate system, also referred to as the discrete mass model and the distributed mass model respectively. The discrete co-ordinate system defines moments and angles of torsion at a set of discrete points in terms of components having specified directions. The analytical procedure for this type can be greatly simplified as an eigenvalue problem because the inertia moments are developed only at these nodal points. In beam structures, the lumped mass matrix [1, 2] is derived as a diagonal matrix by applying one-half of the total rotational mass to each nodal point. Moreover, the mass influence coefficients are determined by using the cubic Hermitian polynomials as the shape functions. This result is called the consistent mass matrix [3, 4] and it contains many off-diagonal terms due to the effects coupling. Krajcinovic [5] presented a consistent mass matrix by using hyperbolic functions which are partial solutions of homogeneous differential equations governing the static torsional problem.

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The second type of mass system to be distinguished is the distributed co-ordinate system [6, 7]. The structural properties of mass and stiffness are generally distributed continuously throughout a beam. The ultimate idealized model is one which would reflect the continuity of these parameters. In this case, the dynamic stiffness equation of a beam segment subjected to torsional vibration is established by using a general solution of the differential equation of motion. This distributed mass model leads to an eigenvalue problem of a transcendental equation, and gives an exact solution. This approach is called the eigenstiffness matrix method [8], or the continuous mass method [9]. In the last four years Banerjee *et al.* [10] studied the dynamic stiffness matrix of continuous beams with thin-walled cross-section. But the influence coefficients of the dynamic member stiffness matrix have not been derived in a closed form.

In the present study, a procedure for natural vibration analysis is elucidated using the exact method [1] based on a general solution of a differential equation of motion. The calculated results regarding natural frequencies are compared with the approximate method based on the discrete co-ordinate system. The effect of the number of beam elements and the accuracy of natural frequencies obtained by three different mass matrix methods are investigated. Lastly, several terms derived from a Taylor's series expansion of the coefficients in the new dynamic stiffness matrix are shown, and the mathematical relationship between the lumped, consistent and continuous mass methods is discussed. The relative relationship between the exact and the approximate methods is demonstrated clearly in the light of the above-mentioned numerical and mathematical considerations.

2. NATURAL VIBRATION ANALYSIS

2.1. DISTRIBUTED CO-ORDINATE SYSTEM

The beam considered in this study is assumed to be straight, having a thin-walled constant cross-section. The axial co-ordinate x coincides with the beam axis through the center of each cross-section. The angle of rotation about the beam axis is denoted by θ. For the sake of simplicity, a thin-walled beam of span length L having a double-symmetrical cross-section is considered, although other cross-sections can also be analyzed. The mathematical formulation of the problem of free torsional vibration is presented [3, 11] by a partial differential equation that is uncoupled if the center of the cross-section is considered with the shear center. The problem is formulated by

$$EI_w \frac{\partial^4\theta}{\partial x^4} - GJ\frac{\partial^2\theta}{\partial x^2} + mr^2\frac{\partial^2\theta}{\partial t^2} = 0, \tag{1}$$

where EI_w is the torsional rigidity associated with warping, GJ is the Saint–Venant torsional rigidity, m is the mass of the beam per unit length, r is the radius of gyration of the cross-section, and t is the time. The solution of equation (1) can be obtained by assuming the free vibration motion to be harmonic

$$\theta(x, t) = \Theta(x)\exp(\mathrm{i}\,\omega\, t), \tag{2}$$

where $\Theta(x)$ is the amplitude of torsional angle at point x for the vibrating beam (eigenfunction), ω is the natural circular frequency, and $\mathrm{i} = \sqrt{-1}$. By substituting equation (2) into equation (1), the eigenfunction $\Theta(x)$ for a thin-walled beam may be written in the following form:

$$\Theta(x) = C_1\cos\mu x + C_2\sin\mu x + C_3\cosh\nu x + C_4\sinh\nu x, \tag{3}$$

in which

$$\mu = \sqrt{\frac{GJ}{2EI_w}(\zeta - 1)}, \quad v = \sqrt{\frac{GJ}{2EI_w}(\zeta + 1)}, \quad \zeta = \sqrt{1 + \frac{4mr^2 EI_w}{(GJ)^2}\omega^2}, \tag{4a–c}$$

and the integration constants C_1, C_2, C_3 and C_4 are determined by the boundary conditions of the continuous beams.

The force quantities (torsional moment $= M_x(x)$ and warping moment $= M_w(x)$) at any point x of a beam are given by Vlasov [11]:

$$M_x(x) = -EI_w\Theta'''(x) + GJ\Theta'(x), \tag{5a}$$

$$M_w(x) = -EI_w\Theta''(x), \tag{5b}$$

where the superscript prime indicates differentiation with respect to the axial co-ordinate x. To obtain the dynamic stiffness matrix expressed by the distributed co-ordinate system for the torsional vibration of the beam segment, the boundary conditions at both ends of the beam (①: $x = 0$ and ②: $x = L$ as shown in Figure 1) are imposed from equation (3)

$$\theta_{x1} = \Theta(0), \qquad \theta_{x2} = \Theta(L), \qquad \theta_{w1} = \Theta'(0), \qquad \theta_{w2} = \Theta'(L),$$

$$M_{x1} = -EI_w\Theta'''(0) + GJ\Theta'(0), \qquad M_{x2} = -EI_w\Theta'''(L) + GJ\Theta'(L),$$

$$M_{w1} = -EI_w\Theta''(0), \qquad M_{w2} = -EI_w\Theta''(L). \tag{6a–h}$$

In the above equations (6), θ_{x1}, θ_{x2} and θ_{w1}, θ_{w2} are, respectively, the torsional angle (the angle of torsion) and the warping torsional angle (the angle of torsion per unit length) at the ends of the beam, while M_{x1}, M_{x2} and M_{w1}, M_{w2} correspond to the torsional and warping moments at these nodal co-ordinates. The end moments and displacements of a beam segment are indicated in Figure 1. By means of the eigenfunction in equation (3), the state vector $\{\theta_{x1}, \theta_{w1}, M_{x1}, M_{w1}\}^T$ of the beam at the point $x = 0$ is related to the integration constant vector $\{C_1, C_2, C_3, C_4\}^T$, and the end displacements (θ_{x2}, θ_{w2}) and end moments (M_{x2}, M_{w2}) of the beam at the point $x = L$ may be expressed in terms of the same integration constant vector. By eliminating the integration constant vector from the above two equations, the final form of the dynamic stiffness matrix relating end harmonic

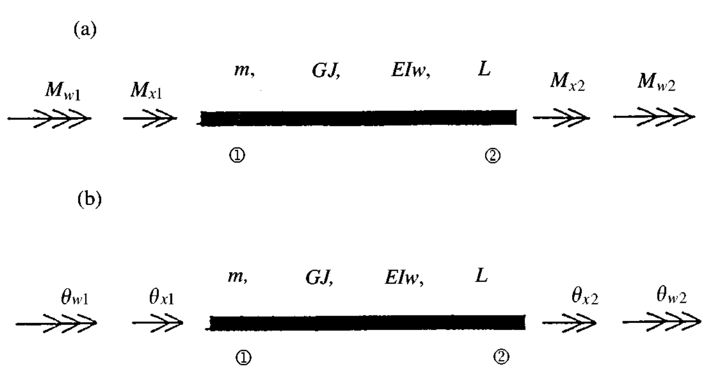

Figure 1. Sign of dynamic stiffness matrix: (a) end torsional moments; (b) end torsional angles.

moments and displacements is obtained by the following matrix notation:

$$\begin{Bmatrix} M_{x1} \\ M_{w1} \\ M_{x2} \\ M_{w2} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ & k_{22} & k_{23} & k_{24} \\ & & k_{33} & k_{34} \\ Sym. & & & k_{44} \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{w1} \\ \theta_{x2} \\ \theta_{w2} \end{Bmatrix}, \tag{7a}$$

or, in abbreviated form

$$\mathbf{F} = \mathbf{K}(\omega)\mathbf{U}, \tag{7b}$$

where **F** is the end force vector, and **U** is the end displacement vector. As for the coefficients of the dynamic stiffness matrix $\mathbf{K}(\omega)$:

$$k_{11} = K(\mu^2 + v^2)(vcS + \mu sC), \quad k_{12} = K\{(\mu^2 - v^2)(1 - cC) + 2\mu vsS\},$$

$$k_{13} = -K(\mu^2 + v^2)(\mu s + vS), \quad k_{14} = K(\mu^2 + v^2)(C - c),$$

$$k_{22} = K(\mu^2 + v^2)(sC/\mu - cS/v), \quad k_{24} = K(\mu^2 + v^2)(S/v - s/\mu),$$

$$k_{23} = -k_{14}, \quad k_{33} = k_{11}, \quad k_{34} = -k_{12}, \quad k_{44} = k_{22},$$

$$K = \frac{EI_w \mu v}{2\mu v(1 - cC) + (v^2 - \mu^2)sS}. \tag{8a–k}$$

in the above, the letters c, s, C and S denote

$$c = \cos \mu L, \quad s = \sin \mu L, \quad C = \cosh vL, \quad S = \sinh vL. \tag{9a–d}$$

The square matrix $\mathbf{K}(\omega)$ obtained in equation (7b) is designated in this study as the eigenstiffness matrix [8] because this matrix $\mathbf{K}(\omega)$ includes the eigenvalues μL and vL of the beam. The eigenstiffness matrix can be used to assemble the system dynamic stiffness matrix for a continuous beam or a space frame in a manner entirely analogous to the assemblage of the system static stiffness matrix from element stiffness matrices. Therefore, the frequency equation of continuous beams or frames for the distributed co-ordinate system can be expressed by means of the principle of superposition

$$\det|\mathbf{K}(\omega)| = 0. \tag{10}$$

This is a transcendental equation of trigonometric and hyperbolic functions which contains the natural circular frequencies ω of the beam. The roots of equation (10) may be obtained by applying the Regula–Falsi method [12] and by using a high-speed digital computer.

Corresponding to a particular value of natural frequency ω_n, an eigenfunction is defined by equation (3) which is also called the modal shape function. This eigenfunction represents the mode shape of the torsionally vibrating beam at each of the eigenvalues. The eigenfunction for continuous beams, as shown in Figure 2, is given in the following equation from equation (3):

$$\Theta_{ni}(x_i) = C_{1ni} \cos \mu_{ni} x_i + C_{2ni} \sin \mu_{ni} x_i + C_{3ni} \cosh v_{ni} x_i + C_{4ni} \sinh v_{ni} x_i, \tag{11}$$

where $\Theta_{ni}(x_i)$ is the eigenfunction of the ith span for the nth modes, and C_{1ni}, C_{2ni}, C_{3ni} and C_{4ni} are the unknown integration constants. The origin of the x_i co-ordinates of continuous

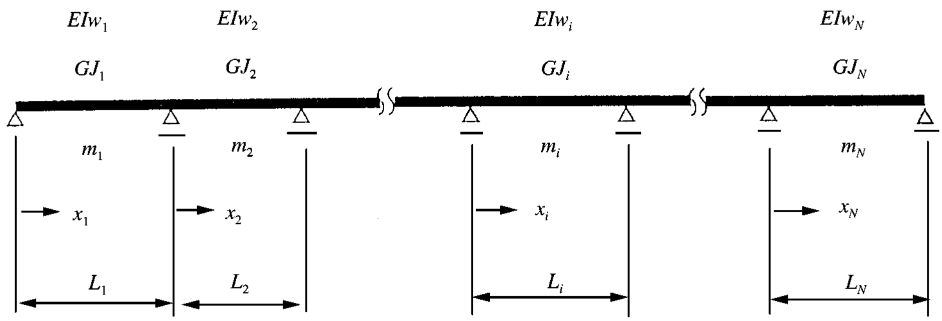

Figure 2. N-span continuous beam.

beams is assumed to be at the left-hand end of each span. By means of the first four expressions in equation (6), the integration constants may be expressed in terms of the displacements at both ends of the beam segment

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{w1} \\ \theta_{x2} \\ \theta_{w2} \end{Bmatrix}, \tag{12a}$$

or, in abbreviated form

$$\mathbf{C} = \mathbf{RU}, \tag{12b}$$

where $\mathbf{C}$ is the integration constant vector and

$$r_{11} = R(cC - 1 - vsS/\mu), \quad r_{12} = R(cS/v - sC/\mu), \quad r_{13} = R(C - c),$$

$$r_{14} = R(s/\mu - S/v), \quad r_{21} = R(sC + vcS/\mu), \quad r_{22} = R\{sS/v - (1 - cC)/\mu\},$$

$$r_{23} = -R(s + vS/\mu), \quad r_{24} = -r_{13}/\mu,$$

$$r_{31} = -\mu r_{22}, \quad r_{32} = -r_{12}, \quad r_{33} = -r_{13}, \quad r_{34} = -r_{14},$$

$$r_{41} = -\mu r_{21}/v, \quad r_{42} = -r_{11}/v, \quad r_{43} = -\mu r_{23}/v, \quad r_{44} = r_{13}/v,$$

$$R = \frac{\mu v}{2\mu v(1 - cC) + (v^2 - \mu^2)sS}. \tag{13a–q}$$

The square matrix $\mathbf{R}$ of order 4×4 in equation (12b) is called in this study the integration constant matrix. The sign used in equation (12) is for a typical beam, as shown in Figure 1, in which both end displacements (θ_{x1}, θ_{w1}, θ_{x2}, θ_{w2}) shown are positive. The values of the integration constants C_{1ni}, C_{2ni}, C_{3ni} and C_{4ni} are determined by substituting both end displacements of individual element beam members into equation (12).

2.2. DISCRETE CO-ORDINATE SYSTEM

In the discrete co-ordinate system, the dynamic stiffness matrix for the torsional action of thin-walled beams is expressed as a superposition of elastic and inertial forces forming, respectively, the stiffness and mass matrices. These matrices are obtained from formulations based on the cubic displacement function that is standard in finite element beam theory [2, 13, 14]. The results are simpler in form than equations (7) and (8) but are less accurate, because the assumed cubic field is only approximate. This is discussed in a later numerical example and power series expansion of the dynamic stiffness matrix developed in this study. The static stiffness matrix $\mathbf{K}_s$ for a thin-walled beam with a constant and open cross-section is given by [3, 15]

$$\mathbf{K}_s = \begin{bmatrix} \frac{12EI_w}{L^3}+\frac{6GJ}{5L} & \frac{6EI_w}{L^2}+\frac{GJ}{10} & -\frac{12EI_w}{L^3}-\frac{6GJ}{5L} & \frac{6EI_w}{L^2}+\frac{GJ}{10} \\ & \frac{4EI_w}{L}+\frac{2GJL}{15} & -\frac{6EI_w}{L^2}-\frac{GJ}{10} & \frac{2EI_w}{L}-\frac{GJL}{30} \\ & & \frac{12EI_w}{L^3}+\frac{6GJ}{5L} & -\frac{6EI_w}{L^2}-\frac{GJ}{10} \\ Sym. & & & \frac{4EI_w}{L}+\frac{2GJL}{15} \end{bmatrix}. \tag{14}$$

By assuming the same displacement functions which are used for formulation of the static stiffness matrix $\mathbf{K}_s$, the consistent mass matrix [3, 4] corresponding to the torsional effects is

$$\mathbf{M}_c = \frac{mr^2L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ Sym. & & & 4L^2 \end{bmatrix} \tag{15}$$

and by applying one-half of the total rotational mass to both nodal points of the beam segment, the lumped torsional mass matrix is obtained [1, 2] as

$$\mathbf{M}_l = mr^2L/24\cdot\{\mathrm{Diag}(12 \quad L^2 \quad 12 \quad L^2)\}, \tag{16}$$

where Diag( ) is the diagonal matrix. This mass matrix $\mathbf{M}_l$ is the simplest method for considering the inertial properties of a dynamic system.

The static stiffness and mass matrices for each beam element are assembled into system stiffness matrix $\mathbf{K}$ and system mass matrix $\mathbf{M}$ for the entire beam. To avoid singularities in an eigenvalue solution, the boundary conditions for the beam are considered by fully elimination of all fixed degree of freedom (dof.) The frequency equation of the discrete co-ordinate system (lumped and consistent mass methods) may be given as follows:

$$\det|\mathbf{K}-\omega^2\mathbf{M}| = 0. \tag{17}$$

Equation (17) is an important mathematical problem known as an eigenvalue problem. There are several computing methods that are widely used for solving the eigenvalue problem of vibration systems. The system stiffness and mass matrices in equation (17) are the symmetric matrix. The Householder method for determining the eigenvalues and eigenvectors of a symmetric matrix is used in this study. A sequence of Householder transformations that reduce the symmetric matrix to a simpler tri-diagonal form is more efficient for analysis of large-sized eigenvalue problems [16, 17]. The computation of the eigenvalues in the present analysis is carried out through a Householder-Bisection-Inverse Iteration Solution subroutine (DEIGAB and DEIGRS), a double-precision version of which is available from the mathematical subprogram package IMSL at the Hokkaido University Computing Center.

3. NUMERICAL RESULTS

A numerical example is presented to demonstrate the applicability of the analytical method developed here and to delineate some characteristics of torsional vibration of continuous beams. Figure 3 shows a single-span beam and two- and three-span continuous beams with a thin-walled uniform cross-section. The geometry of the beams and the structural properties necessary for natural vibration analysis are given as follows: span length, $L = 31{\cdot}5$ m, warping stiffness, $EI_w = 1{\cdot}336 \times 10^{10}$ N m^4, torsional stiffness, $GJ = 2{\cdot}789 \times 10^{10}$ N m^2, polar moment of inertia, $I_p = 1{\cdot}1023$ m^4, cross-sectional area, $A = 0{\cdot}2033$ m^2, and dead load of the beam per unit length, $w = 1{\cdot}566 \times 10^4$ N/m. All data on the geometry and structural properties of beams are the same, and the beams are subdivided into N beam segments of equal length. The boundary conditions of a single-span beam and continuous beams are restrained against the angle of torsion about the beam axis with no restraint of warping.

The calculated results of the single-span beam and two- and three-span continuous beams by the lumped and consistent mass methods are presented for the first 10 modes in Tables 1, 2 and 3, respectively, and are compared with the exact solutions obtained by the continuous mass method. Also, the percentage differences between the exact solutions due to the continuous mass method and the approximate natural frequencies due to the lumped and consistent mass methods are given in parentheses in these tables. By means of both the lumped and consistent mass methods, the values of approximate natural frequencies gradually approach the exact solutions as the number N of beam segments increases. In general, the torsional natural frequencies obtained by the use of the lumped and consistent mass methods are the lower and upper bounds to the exact solutions respectively. It may be also pointed out that for the same number of beam segments, the use of the consistent mass method provides a higher level of accuracy than that of the lumped mass method.

It is seen in Tables 1–3 that there is an interesting pattern in which the natural frequencies calculated by the lumped and consistent mass methods fall into some groups, with as many modes in each group as the number of spans in the continuous beam. The grouping of the natural frequencies and the percentage differences from the exact solution are repeated at constant intervals according to the number of spans. The first, second, third and fourth

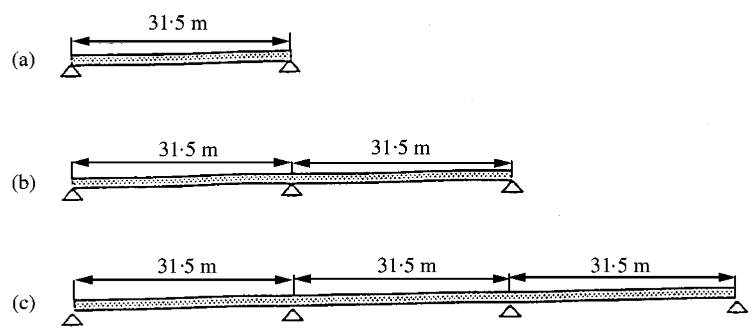

Figure 3. Numerical examples: (a) single-span beam; (b) two-span continuous beam; (c) three-span continuous beam.

TABLE 1

Torsional natural frequencies (Hz) of single-span beam

Mode order	Lumped mass method				Consistent mass method				Continuous mass method
	$N=6$	$N=8$	$N=10$	$N=12$	$N=6$	$N=8$	$N=10$	$N=12$	
1	28·273 – (1·136)	28·415 – (0·640)	28·481 – (0·410)	28·517 – (0·285)	28·598 (0·000)	28·598 (0·000)	28·598 (0·000)	28·598 (0·000)	28·598
2	55·030 – (4·466)	56·143 – (2·534)	56·665 – (1·629)	56·950 – (1·134)	57·605 (0·003)	57·603 (0·001)	57·603 (0·000)	57·603 (0·000)	57·603
3	78·870 – (9·771)	82·516 – (5·600)	84·246 – (3·621)	85·200 – (2·529)	87·438 (0·031)	87·419 (0·009)	87·414 (0·003)	87·412 (0·001)	87·411
4	98·596 – (16·728)	106·899 – (9·715)	110·910 – (6·327)	113·145 – (4·440)	118·573 (0·144)	118·453 (0·043)	118·422 (0·016)	118·411 (0·007)	118·402
5	113·140 – (25·038)	128·713 – (14·720)	136·330 – (9·673)	140·626 – (6·827)	151·613 (0·453)	151·147 (0·144)	151·016 (0·057)	150·970 (0·026)	150·930
6		147·423 – (20·449)	160·173 – (13·569)	167·454 – (9·640)		186·011 (0·373)	185·603 (0·153)	185·453 (0·073)	185·319
7		162·359 – (26·819)	182·107 – (17·918)	193·411 – (12·823)		223·651 (0·807)	222·625 (0·345)	222·230 (0·167)	221·860
8			201·777 – (22·635)	218·258 – (16·316)			262·581 (0·678)	261·688 (0·335)	260·813
9			218·589 – (27·716)	241·735 – (20·062)			305·992 (1·187)	304·245 (0·609)	302·403
10				263·535 – (24·015)				350·351 (1·017)	346·825

TABLE 2

Torsional natural frequencies (Hz) of two-span continuous beam

Mode order	Lumped mass method				Consistent mass method				Continuous mass method
	$N = 12$	$N = 16$	$N = 20$	$N = 24$	$N = 12$	$N = 16$	$N = 20$	$N = 24$	
1	28·273	28·415	28·481	28·517	28·598	28·598	28·598	28·598	28·598
	− (1·136)	− (0·640)	− (0·410)	− (0·285)	(0·000)	(0·000)	(0·000)	(0·000)	
2	29·084	29·150	29·178	29·193	29·436	29·346	29·303	29·279	29·242
	− (0·539)	− (0·313)	− (0·218)	− (0·166)	(0·665)	(0·356)	(0·208)	(0·128)	
3	55·030	56·143	56·665	56·950	57·605	57·603	57·603	57·603	57·603
	− (4·466)	− (2·534)	− (1·629)	− (1·134)	(0·003)	(0·001)	(0·000)	(0·000)	
4	56·500	57·550	58·028	58·287	59·296	59·111	59·024	58·977	58·902
	− (4·078)	− (2·295)	− (1·484)	− (1·044)	(0·669)	(0·355)	(0·207)	(0·128)	
5	78·870	82·516	84·246	85·200	87·438	87·419	87·414	87·412	87·411
	− (9·771)	− (5·600)	− (3·621)	− (2·529)	(0·031)	(0·009)	(0·003)	(0·001)	
6	80·716	84·474	86·215	87·167	90·045	89·718	89·576	89·503	89·387
	− (9·701)	− (5·497)	− (3·549)	− (2·484)	(0·736)	(0·370)	(0·211)	(0·130)	
7	98·596	106·899	110·910	113·145	118·573	118·453	118·422	118·411	118·402
	− (16·728)	− (9·715)	− (6·327)	− (4·440)	(0·144)	(0·043)	(0·016)	(0·007)	
8	100·446	109·247	113·400	115·693	122·242	121·610	121·368	121·253	121·085
	− (17·045)	− (9·777)	− (6·347)	− (4·453)	(0·956)	(0·434)	(0·234)	(0·139)	
9	113·140	128·713	136·330	140·626	151·613	151·147	151·016	150·970	150·930
	− (25·038)	− (14·720)	− (9·673)	− (6·827)	(0·453)	(0·144)	(0·057)	(0·026)	
10	114·565	131·273	139·236	143·696	156·546	155·272	154·811	154·612	154·354
	− (25·778)	− (14·954)	− (9·794)	− (6·905)	(1·420)	(0·594)	(0·296)	(0·167)	

TABLE 3

Torsional natural frequencies (Hz) of three-span continuous beam

Mode order	Lumped mass method				Consistent mass method				Continuous mass method
	$N = 18$	$N = 24$	$N = 30$	$N = 36$	$N = 18$	$N = 24$	$N = 30$	$N = 36$	
1	28·273	28·415	28·481	28·517	28·598	28·598	28·598	28·598	28·598
	– (1·136)	– (0·640)	– (0·410)	– (0·285)	(0·000)	(0·000)	(0·000)	(0·000)	
2	28·674	28·778	28·826	28·851	29·012	28·968	28·946	28·935	28·917
	– (0·841)	– (0·478)	– (0·315)	– (0·226)	(0·328)	(0·176)	(0·103)	(0·064)	
3	29·507	29·532	29·539	29·544	29·873	29·735	29·668	29·632	29·575
	– (0·229)	– (0·144)	– (0·119)	– (0·105)	(1·010)	(0·541)	(0·315)	(0·194)	
4	55·030	56·143	56·665	56·950	57·605	57·603	57·603	57·603	57·603
	– (4·466)	– (2·534)	– (1·629)	– (1·134)	(0·003)	(0·001)	(0·000)	(0·000)	
5	55·759	56·841	57·340	57·612	58·442	58·350	58·307	58·284	58·247
	– (4·271)	– (2·414)	– (1·556)	– (1·089)	(0·334)	(0·177)	(0·103)	(0·064)	
6	57·259	58·277	58·732	58·977	60·175	59·892	59·758	59·687	59·572
	– (3·882)	– (2·174)	– (1·410)	– (0·998)	(1·013)	(0·537)	(0·313)	(0·193)	
7	78·870	82·516	84·246	85·200	87·438	87·419	87·414	87·412	87·411
	– (9·771)	– (5·600)	– (3·621)	– (2·529)	(0·031)	(0·009)	(0·003)	(0.001)	
8	79·793	83·491	85·226	86·179	88·734	88·562	88·489	88·452	88·394
	– (9·730)	– (5·546)	– (3·583)	– (2·506)	(0·385)	(0·190)	(0·108)	(0·066)	
9	81·654	85·477	87·226	88·176	91·396	90·904	90·689	90·578	90·402
	– (9·676)	– (5·448)	– (3·513)	– (2·462)	(1·099)	(0·555)	(0·317)	(0·195)	
10	98·596	106·899	110·910	113·145	118·573	118·453	118·422	118·411	118·402
	– (16·728)	– (9·715)	– (6·327)	– (4·440)	(0·144)	(0·043)	(0·016)	(0·007)	

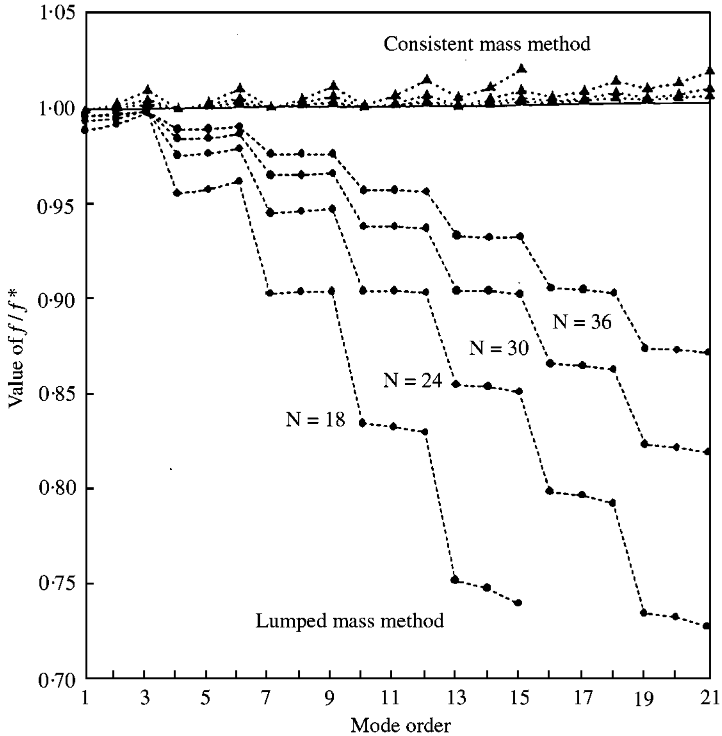

Figure 4. Natural frequency ratio f/f^* of three-span continuous beam.

modes of the single-span beam correspond to the first, third, fifth and seventh modes of the two-span continuous beam, and to the first, fourth, seventh and 10th modes of the three-span continuous beam respectively.

Figure 4 indicates the relationship between the natural frequency ratio f/f^* and the order of natural modes for the three-span continuous beam. Here, f^* is the exact solution obtained by the continuous mass method, and f is the approximate solution obtained by using the lumped and consistent mass methods. The values of natural frequencies calculated by using the lumped mass method are considerably small in comparison with those of the exact solutions, as the mode order is higher. On the other hand, the values of natural frequencies obtained by the consistent mass method are a little large in comparison with the exact solutions. The torsional natural frequencies of continuous beams obtained by the consistent mass method can be computed more accurately than those of the lumped mass method, for the same required order of natural modes. It may be confirmed from Figure 4 that there is a mode grouping in torsional natural frequencies of three-span continuous beams calculated by the lumped mass method.

4. POWER SERIES EXPANSION

The mathematical relationship between the lumped, consistent and continuous mass methods is established in this study. It is shown that the terms of the power series expansion of the coefficients in the eigenstiffness matrix of equation (8) are precisely the stiffness and mass matrices in common use in the finite element method (displacement method). For simplicity, the dynamic coefficient k_{11} of equation (8a) is expanded in a series. It can be derived from a Taylor's series expansion as follows:

$$k_{11} = \frac{EI_w \mu\nu(\mu^2 + \nu^2)(\nu\cos\mu L \sinh\nu L + \mu\sin\mu L\cosh\nu L)}{2\mu\nu(1 - \cos\mu L\cosh\nu L) + (\nu^2 - \mu^2)\sin\mu L\sinh\nu L}$$
$$= \frac{12EI_w}{L^3} + \frac{6GJ}{5L} - \frac{13}{35}mr^2L\omega^2 - \frac{GJL}{700}\left(\frac{GJ}{EI_w}\right) + \frac{mr^2L^3}{3150}\left(\frac{GJ}{EI_w}\right)\omega^2 + \frac{GJL^3}{63\,000}\left(\frac{GJ}{EI_w}\right)^2 + \cdots. \tag{18}$$

It should be recognized that the first two terms on the right-hand side of equation (18) are equal to the corresponding stiffness coefficient of the matrix in equation (14), and the third term is equal to the consistent mass coefficient of the matrix in equation (15). Similarly, the Taylor's series expansions of the dynamic stiffness coefficients of matrix for torsional vibration may be written in matrix notation as

$$\mathbf{K}(\omega) = \mathbf{K}_0 \cdot EI_w + \mathbf{G}_0 \cdot GJ - \mathbf{M}_0 \cdot \omega^2 - \mathbf{G}_1 \cdot (GJ)^2 + \mathbf{M}_1 \cdot GJ\omega^2 + \mathbf{G}_2 \cdot (GJ)^3 + \cdots, \tag{19}$$

where the coefficients of the first six matrices in the above equation (19) are given in Appendix A.

The first two matrices ($\mathbf{K}_0 \cdot EI_w + \mathbf{G}_0 \cdot GJ$) agree precisely with the static stiffness matrix $\mathbf{K}_s$ shown in equation (14), and the third matrix $\mathbf{M}_0$ is also equal to the consistent mass matrix $\mathbf{M}_c$ in equation (15). The other matrices $\mathbf{G}_1$, $\mathbf{M}_1$ and $\mathbf{G}_2$ in equation (19) correspond to higher order terms obtained by expanding the dynamic stiffness matrix in Taylor's series expansions. It is seen that the values of these coefficients of matrices $\mathbf{G}_1$, $\mathbf{M}_1$ and $\mathbf{G}_2$ are considerably small in comparison with those of matrices $\mathbf{K}_0$, $\mathbf{G}_0$ and $\mathbf{M}_0$. Therefore, it is concluded that the consistent mass method applies in a special case of the continuous mass method obtained by neglecting higher order terms. Moreover, it can be estimated that the lumped mass method is a truncated result of the consistent mass method which is obtained by omitting the mass coupling. The relative relationship between the exact and approximate mass methods is demonstrated clearly from the aforementioned mathematical consideration, and it is also easily comprehensible from the computed results shown in Tables 1–3.

5. CONCLUSIONS

An analytical method based on the general solution of a differential equation of motion for torsional vibration of beams with thin-walled cross-sections is developed in this study. The coefficients of the dynamic stiffness matrix are given a closed form, and can be used in assembling the dynamic stiffness matrix for the entire structure by using the same procedure employed in assembling the stiffness and mass matrices for the discrete co-ordinate system. The mathematical relationship between the exact method and the approximate method based on a finite element approach is established. The static stiffness matrix and consistent mass matrix in equations (14) and (15) are represented by the first three terms of a power-series expansion of the coefficients of the dynamic stiffness matrix of equation (7). Consequently, it is concluded that the consistent mass method is a special procedure of the continuous mass method which disregards the effects of higher order terms. Moreover, the lumped mass method is a truncated result of the consistent mass method which is obtained by omitting the mass coupling in order to simplify the problem.

It is seen from Tables 1–3 that the values of natural frequencies obtained by the consistent mass method are the upper bounds to the exact solution. The conventional lumped mass method yields natural frequencies that are considerably lower than the exact solution for the same number of beam segments. In general, the torsional natural frequencies of continuous beams can be calculated more accurately by the consistent mass method than by the lumped mass method. The torsional natural frequencies of continuous beams with a uniform span length have an interesting group of modes corresponding to the number of spans. It is also confirmed that an aberration to the exact solution is repeated in the constant according to the number of spans. This study provides a basis for future theoretical research and may be applied to structural dynamics concerning the triple coupling vibration of thin-walled straight continuous beams and horizontally curved beams with inclusion of the effects of warping torsion.

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APPENDIX A: MATRICES OF EQUATION (19)

The coefficients of matrices $\mathbf{K}_0$, $\mathbf{G}_0$, $\mathbf{M}_0$, $\mathbf{G}_1$, $\mathbf{M}_1$ and $\mathbf{G}_2$ in equation (19) are represented as follows. The first matrix (stiffness matrix) $\mathbf{K}_0$ is

$$\mathbf{K}_0 = \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ & & \frac{12}{L^3} & -\frac{6}{L^2} \\ Sym. & & & \frac{4}{L} \end{bmatrix}. \tag{A1}$$

The second matrix (geometrical matrix) $\mathbf{G}_0$ is

$$\mathbf{G}_0 = \begin{bmatrix} \frac{6}{5L} & \frac{1}{10} & -\frac{6}{5L} & \frac{1}{10} \\ & \frac{2L}{15} & -\frac{1}{10} & -\frac{L}{30} \\ & & \frac{6}{5L} & -\frac{1}{10} \\ Sym. & & & \frac{2L}{15} \end{bmatrix}. \tag{A2}$$

The third matrix (mass matrix) $\mathbf{M}_0$ is

$$\mathbf{M}_0 = mr^2L \begin{bmatrix} \frac{13}{35} & \frac{11L}{210} & \frac{9}{70} & -\frac{13L}{420} \\ & \frac{L^2}{105} & \frac{13L}{420} & -\frac{L^2}{140} \\ & & \frac{13}{35} & -\frac{11L}{210} \\ Sym. & & & \frac{L^2}{105} \end{bmatrix}. \tag{A3}$$

The fourth matrix (second order geometrical matrix) $\mathbf{G}_1$ is

$$\mathbf{G}_1 = \frac{L}{EI_w} \begin{bmatrix} \frac{1}{700} & \frac{L}{1400} & -\frac{1}{700} & \frac{L}{1400} \\ & \frac{11L^2}{6300} & -\frac{L}{1400} & -\frac{13L^2}{12\,600} \\ & & \frac{1}{700} & -\frac{L}{1400} \\ Sym. & & & \frac{11L^2}{6300} \end{bmatrix}. \tag{A4}$$

The fifth matrix (second order mass-geometrical matrix) $\mathbf{M}_1$ is

$$\mathbf{M}_1 = \frac{mr^2L^3}{EI_w}\begin{bmatrix} \frac{1}{3150} & \frac{L}{1260} & -\frac{1}{3150} & -\frac{L}{1680} \\ & \frac{L^2}{3150} & \frac{L}{1680} & -\frac{L^2}{3600} \\ & & \frac{1}{3150} & -\frac{L}{1260} \\ Sym. & & & \frac{L^2}{3150} \end{bmatrix}. \tag{A5}$$

Finally, the sixth matrix (third order geometrical matrix) $\mathbf{G}_2$ is

$$\mathbf{G}_2 = \frac{L^3}{100(EI_w)^2}\begin{bmatrix} \frac{1}{630} & \frac{L}{1260} & -\frac{1}{630} & \frac{L}{1260} \\ & \frac{L^2}{270} & -\frac{L}{1260} & -\frac{L^2}{378} \\ & & \frac{1}{630} & -\frac{L}{1260} \\ Sym. & & & \frac{L^2}{270} \end{bmatrix}. \tag{A6}$$

APPENDIX B: NOMENCLATURE

A	cross-sectional area
C_1, C_2, C_3, C_4	integration constants
$\mathbf{C}$	integration constant vector
EI_w	warping rigidity
f, f^*	natural frequencies
$\mathbf{F}$	end force vector
GJ	torsional rigidity
$\mathbf{G}_0$, $\mathbf{G}_1$, $\mathbf{G}_2$	coefficient matrices
I_p	polar moment of inertia
k_{ij}	coefficients of dynamic stiffness matrix
K	coefficient
$\mathbf{K}$, $\mathbf{K}_0$, $\mathbf{K}_s$	stiffness matrices
$\mathbf{K}(\omega)$	dynamic stiffness matrix
L	span length
M_x, M_w	torsional and warping moments respectively
m	mass of beam per unit length
$\mathbf{M}_c$, $\mathbf{M}_l$	consistent and lumped mass matrices respectively
$\mathbf{M}_0$, $\mathbf{M}_1$	mass matrices
N	number of beam segments
r	radius of gyration of the cross-section
r_{ij}	coefficients of integration constant matrix
R	coefficient
$\mathbf{R}$	integration constant matrix
t	time co-ordinate
$\mathbf{U}$	displacement vector
w	dead load of beam per unit length
x	co-ordinate
θ, θ_x	torsional angles
θ_w	angle of torsion per unit length

$\Theta(x)$	eigenfunction
μ, v, ζ	frequency parameters
ω	natural circular frequency
Diag( )	diagonal matrix
det| |	determinant of matrix
$\{\ \}^T$	transposition matrix
$(')$, $('')$, $(''')$	derivatives with respect to co-ordinate x